On the Dispersion of Slow Rayleigh Fading Channels

Ebrahim MolavianJazi and J. Nicholas Laneman

Motivation

Wireless networks with low-latency requirements are finding numerous applications; such as machine-to-machine **(M2M)** communications.

Finite blocklength results inherently depend upon the CDF of the channel mutual information random variable [1] and its statistics, specially the second order statistic known as the channel dispersion [2,3].

Modified Mutual Information

The mutual information RV

$$i(x^n; y^n) = \log \frac{P_{Y^n|X^n}(y^n|x^n)}{P_{Y^n}(y^n)}$$

with non-i.i.d. input: $i(X^n; Y^n) \neq \sum_{t=1}^n i(X_t; Y_t)$

Fix $Q_{Y^n} = \prod_{t=1}^n Q_{Y_t}$ and define the modified mutual information RV

$$\tilde{i}(x^n; y^n) = \log \frac{P_{Y^n|X^n}(y^n|x^n)}{Q_{Y^n}(y^n)}$$

Then $\tilde{i}(X^n; Y^n) = \sum_{t=1}^n \tilde{i}(X_t; Y_t)$

For Achievability: use modified typicality $\tilde{i}(X^n; Y^n) > \log \gamma_n$

Then, the relevant outage probability is

$$P_{X^n} P_{Y^n|X^n}[\tilde{i}(X^n; Y^n) \leq \log \gamma_n]$$

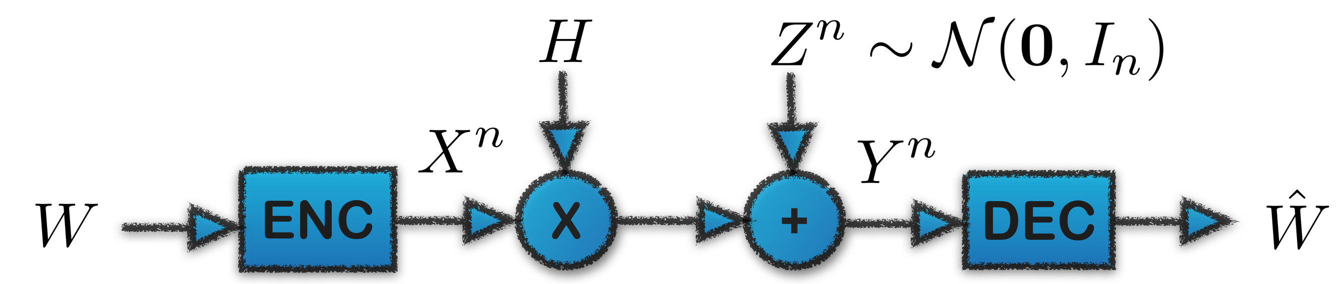
Moreover, a change of measure & uniform bound technique takes care of the non-i.i.d. output distribution.

$$\begin{aligned} & P_{X^n} P_{Y^n}[\tilde{i}(X^n; Y^n) > \log \gamma_n] \\ &= \mathbb{E}_{P_{X^n}}\left[\mathbb{E}_{Q_{Y^n}}\left[\underbrace{\frac{dP_{Y^n}(Y^n)}{dQ_{Y^n}(Y^n)}}_{\leq K,\ \forall y^n} 1\left\{\tilde{i}(X^n; Y^n) > \log \gamma_n\right\}\right]\right] \\ &\leq K P_{X^n} Q_{Y^n}[\tilde{i}(X^n; Y^n) > \log \gamma_n] \end{aligned}$$

For Outer Bound: use another change of measure [4]

$$\begin{aligned} P(D) &= P\left(D \cap \left(\log \frac{dP}{dQ} \leq \log \gamma\right)\right) + P\left(D \cap \left(\log \frac{dP}{dQ} > \log \gamma\right)\right) \\ &\leq \gamma Q\left(D \cap \left(\log \frac{dP}{dQ} \leq \log \gamma\right)\right) + P\left(\log \frac{dP}{dQ} > \log \gamma\right) \\ &\leq \gamma Q(D) + P\left(\tilde{i} > \log \gamma\right) \end{aligned}$$

Problem



Problem: For fixed n, ϵ and given CSIR, maximize M such that

$$W \in \{1, \ldots, M\} \qquad \frac{1}{n}||x^n(w)||^2 \leq P \qquad \Pr[\hat{W} \neq W] \leq \epsilon$$

Non-Asymptotic Bounds

Achievability: For the slow fading channel $P_H P_{Y^n|HX^n}$, fix the input P_{X^n} and the auxiliary output $Q_{Y^n|H}$ satisfying for all y^n, h

$$\frac{dP_{Y^n|H}(y^n|h)}{dQ_{Y^n|H}(y^n|h)} \leq K$$

then codes satisfying constraint $\mathcal{F}_n$ exist with

$$\begin{aligned} \epsilon &\leq P_H P_{X^n} P_{Y^n|HX^n}\left[\tilde{i}(X^n; Y^n|H) \leq \log \gamma_n\right] \\ &\quad + KM P_H P_{X^n} Q_{Y^n|H}\left[\tilde{i}(X^n; Y^n|H) > \log \gamma_n\right] + P_{X^n}[\mathcal{F}_n^c] \end{aligned}$$

where optimal threshold is $\gamma_n = KM$

Converse: For the slow fading channel $P_H P_{Y^n|HX^n}$ and for any auxiliary output $Q_{Y^n|H}$, every valid code satisfies

$$\epsilon \geq P_H P_{X^n} P_{Y^n|HX^n}\left[\tilde{i}(X^n; Y^n|H) \leq \log \gamma_n\right] - \frac{\gamma_n}{M}$$

Second-Order Analysis

Input (Ach.): uniform on power shell [5] $\quad P_{X^n}[\mathcal{F}_n^c] = 0$

Input (Con.): codes on the power shell $\quad ||x_n||^2 = nP$

Auxiliary output: $\quad Q_{Y^n|H}(y^n|h) \sim \mathcal{N}(\mathbf{0}, (1 + Ph^2) I_n)$

$$\frac{dP_{Y^n|H}(y^n|h)}{dQ_{Y^n|H}(y^n|h)} \leq K = O(1)$$

With thresholds $\gamma_n = KM$ for Ach. and $\gamma_n = M/\sqrt{n}$ for Con.

$$\epsilon - \frac{B_1(P)}{\sqrt{n}} \leq \mathbb{E}_H\left[Q\left(\frac{C(PH^2) - \log(KM)/n}{\sqrt{V(PH^2)/n}}\right)\right]$$

$$\epsilon + \frac{B_2(P)}{\sqrt{n}} \geq \mathbb{E}_H\left[Q\left(\frac{C(PH^2) - \log(M/\sqrt{n})/n}{\sqrt{V(PH^2)/n}}\right)\right]$$

where

$$C(P) = \frac{1}{2}\log(1+P) \qquad V(P) = \frac{\log^2 e}{2} \frac{P(P+2)}{(1+P)^2}$$

Define $R^*(n, \epsilon)$ as the solution to $f(R) = \epsilon$ where

$$f(R) := \mathbb{E}_H\left[Q\left(\frac{C(PH^2) - R}{\sqrt{V(PH^2)/n}}\right)\right]$$

- f(R) strictly increasing in R
- perturbation effect: $R^*\left(n, \epsilon + O\left(\frac{1}{\sqrt{n}}\right)\right) = R^*(n, \epsilon) + O\left(\frac{1}{n}\right)$

Theorem: The finite blocklength coding rate of slow fading channels is given by

$$R^*(n, \epsilon) + O\left(\frac{1}{n}\right) \leq \frac{\log M}{n} \leq R^*(n, \epsilon) + \frac{1}{2}\frac{\log n}{n} + O\left(\frac{1}{n}\right)$$

Discussion

- Approach extends to complex noise and general fading
- Outage-capacity as the infinite blocklength performance

$$P_H\left[C(PH^2) < C_{\text{out}}\right] = \epsilon$$

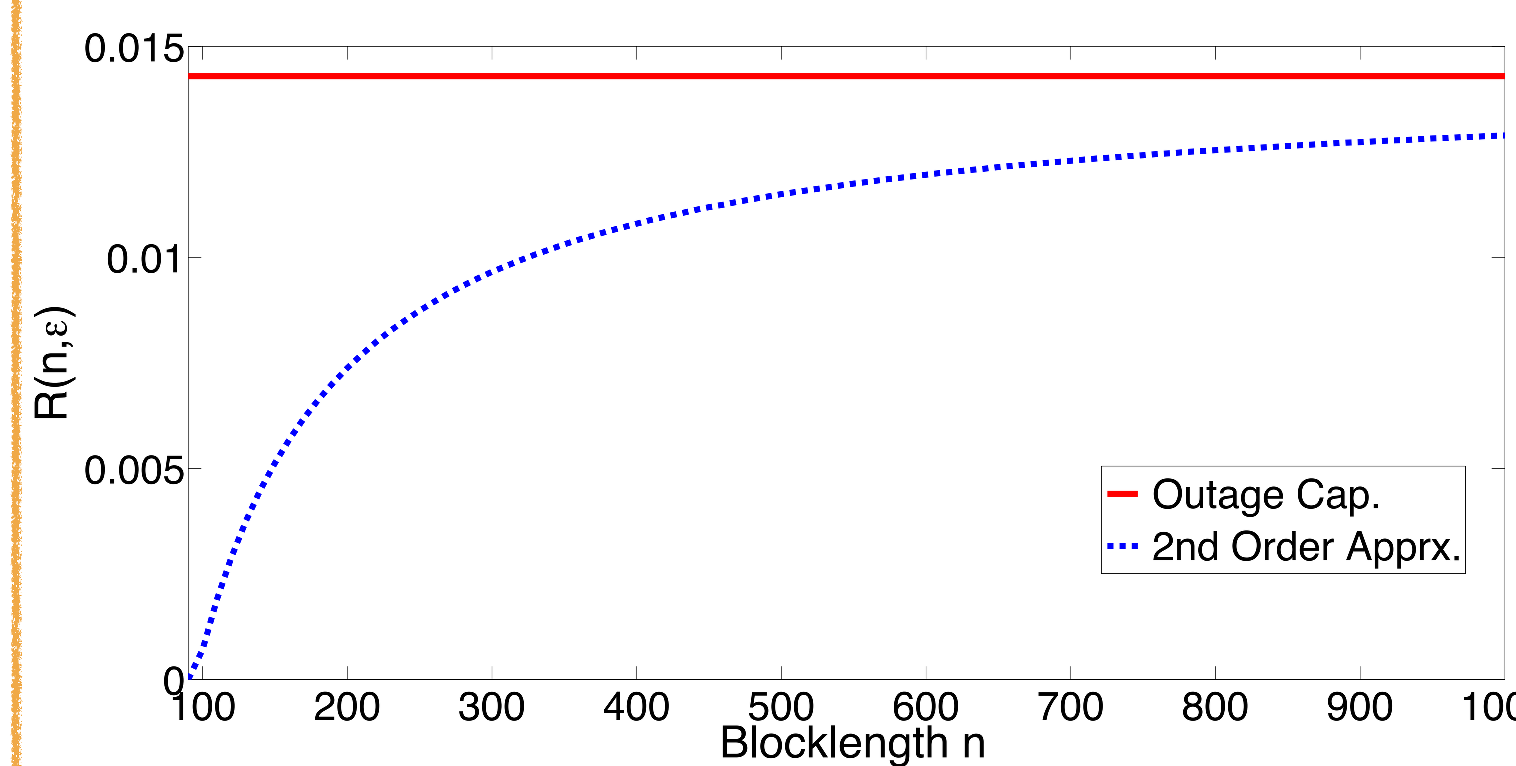
and ours as the finite blocklength behavior

$$\mathbb{E}_H\left[Q\left(\frac{C(PH^2) - R^*(n, \epsilon)}{\sqrt{V(PH^2)/n}}\right)\right] = \epsilon$$

- Concurrent approximation of Wang et al. [6]

$$\frac{\log M}{n} = C_{\text{out}} + O\left(\frac{\log n}{n}\right)$$

not useful since lacks the exact coefficient of the 'log(n)' term



References

[1] S. Verdú and T.-S. Han, **"A General Formula for Channel Capacity,"** Trans. IT, 1994.

[2] Y. Polyanskiy, H. V. Poor and S. Verdú, **"Channel Coding Rate in the Finite Blocklength Regime,"** Trans. IT, 2010.

[3] M. Hayashi, **"Information Spectrum Approach to Second-Order Coding Rate in Channel Coding,"** Trans. IT, 2009.

[4] J. Wolfowitz, **"Notes on a General Strong Converse,"** Information and Control, 1968.

[5] C. E. Shannon, **"Probability of Error for Optimal Codes in a Gaussian Channel,"** Bell Systems Technical Journal, 1959.

[6] W. Yang, G. Durisi, T. Koch and Y. Polyanskiy, **"Quasi-Static SIMO Fading Channels at Finite Blocklength,"** to appear in Proc. ISIT, Istanbul, Turkey, July 2013.

[7] Y. Polyanskiy, H. V. Poor and S. Verdú, **"Dispersion of the Gilbert-Elliott Channel,"** Trans. IT, 2011.